Cycle Analysis – A Case Study

Part 25: Testing Cycles for Statistical Significance*

Applying the Bartels' Test of Significance to a Time Series Cycle Analysis

by CHARLES E. ARMSTRONG

In the cycle analysis of time series, the question almost invariably arises as to whether or not a given period shows statistical evidence of reality. Among the various devices used in testing for significance, Bartels' technique seems to give perhaps the most reasonable results. This test of significance, when applied to the cycle analysis of time series, reflects the degree to which a particular periodicity is consistently present throughout the series under study, as well as the persistence of the cycle, i.e., the number of waves contained in the series. In this test the cycle curve for the given period under test is fitted to the entire series, and the amplitude of the resulting measured cycle is compared with the amplitude which might be expected to occur as a result of chance factors alone. This expected amplitude is determined by fitting the cycle curve for the given period separately to successive segments of the series, each segment being one period in length. From the n individual amplitudes so obtained, an estimate of the expected average amplitude is computed in accordance with the rule used in determining the dispersion of the mean, i.e., by dividing the quadratic mean of the individual amplitudes by $\sqrt{n}$. By comparing the measured average amplitude as obtained by fitting the cycle curve to the entire series with the expected average amplitude, we have a direct means, through the use of standard probability formulae, of arriving at a mathematical measure of the genuineness of the observed cycle. As will be seen, the resulting measure of genuineness will be high in cases where individual cycles exhibit stability in both amplitude and timing, and low where the opposite conditions obtain. Also, the value of n, the number of individual periods contained in the series, has a positive effect on this measure. Of course, for $n=1$ the test is meaningless.

Since periods other than the particular one under analysis affect both the measured and expected average amplitudes in the same way and to the same degree, the final measure of genuineness is not affected by such disturbing elements. Likewise, the effects of any serial correlation present in the series is nullified by appearing in both sides of the ratio of measured to expected amplitude. Hence, the test may with safety be applied to series of deviations from moving averages or from other trend or smoothing curve devices, the use of which may change the amount of serial correlation present in the series.

The Bartels' test is not designed primarily as a means of locating the periods of cycles present in a series. Its chief value lies in its application as a test of significance after the period has been located by some other means. However, in the process of locating the exact period, it is often possible to set up the work processes and papers in such a way as to facilitate the application of the Bartels' test after the period has been determined. The following tables and chart illustrate the application of the test to the results of a harmonic analysis of a typical economic series.

Table I shows the initial and final portions of a monthly series of relative deviations from trend of an index of economic activity in the United States, from 1899 to 1939, inclusive. Investigation has indicated the presence in this series of a fairly marked cycle closely approximating 41 months in duration. The problem is to apply the Bartels' test to this cycle as isolated by methods of harmonic analysis, and so to evaluate the probability that it could be the result of chance conformations in the data.

The first step is to break up the entire series into segments each 41 months in length. Each of these segments, of which there are twelve, is subjected to a standard harmonic analysis for a 41-month period. The method of doing this, for the first and twelfth segments, is shown on Table II[1]. For each of the twelve analyses, the constants A and B are computed as described in Table II, and the twelve sets of values so derived, together with their averages and the sums of their squares, are tabulated in Table III.

Chart I[2] is a scatter diagram depicting the tendency of the twelve sets of A and B values to cluster about their average values. Each pair of A and B values determines a point "p" on the chart. The twelve sets thus determine

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twelve points, p_1, p_2, p_3, ... p_{12}, and the average values determine point "P". The vector Op_1 describes the phase and amplitude[3] of the sinusoid fitted by means of harmonic analysis to the first segment of the series, the vector, Op^2 represents that fitted to the second segment, and so on. Vector OP is the average of the twelve individual vectors, and represents the result of fitting a single 41-month sinusoid to the entire series. The degree to which the individual points p_1, p_2, p_3, ... P_{12}, cluster about P as contrasted to about O is, according to this significance test, indicative of the probable statistical genuineness of the 41-month cycle in this series.

As shown by Table III, the average vector OP has an amplitude of 133.49, and the quadratic mean of the amplitudes of the twelve individual vectors is 203.49. If we assume that there is no real 41-month periodicity in the data under study, and that, therefore, the apparent cyclical movements are the result of chance alone, then the twelve p-points must be considered as random selections from a field of such points distributed about and centered at O. Moreover, if the condition is purely random, the dispersion of all the points in the field about O should not be greatly different from the dispersion of the twelve selected points about O. Under such conditions, the expected amplitude of the single sinusoid fitted to the entire series may be computed as $203.49 \div \sqrt{12} = 58.74$, as contrasted to the measured amplitude to its expectancy is $133.49 \div 58.74 = 2.27$. The probability that this occurs as the result of chance alone is $\frac{1}{e(2.27)^2} = \frac{1}{185}$, where e is the base of natural logarithms.

The illustrative example given here is based on harmonic analysis, but the Bartels' test may be applied to the results of other methods of cycle determination, as well. As in the case described above, the test depends on the relationship between the amplitude of the cycle fitted to the entire series under study and the expected amplitude as derived from the individual cycles fitted to the successive segments of the series.

It should be pointed out that the Bartels' test assumes that genuine periodicities tend towards fixity in both period and amplitude. For, any variations, either systematic or random, in the length of the individual cycles tend to reduce the amplitude of the cycle curve fitted to the entire series through the averaging of out-of-phase ordinates, and so to reduce the final probability that the cycle is genuine. Likewise, variations in amplitude of individual cycles increase the expected average amplitude through the use of quadratic mean, without proportionately increasing the amplitude of the overall cycle curve, and so also reduce the computed probability of genuineness. Hence, for comparability of results, it is generally advisable to express the series, as well as its time scale, if possible, in terms which stabilize to the fullest extent logically consistent with the nature of the series, the amplitude and period of the cycle under test. In this connection, it is usually not feasible to do much with the time scale, but often a change in the form of the series itself will tend to stabilize the cyclical amplitudes involved. For example, in the case used above in illustrating the significance test, the economic data were expressed as relatives to a long-term trend line in preference to absolute deviations, inasmuch as it resulted in a more reasonable approach. Had absolute deviations been used, the final probability would have been of the order of $\frac{1}{165}$.

November, 1944

TABLE I
An Index Of Economic Activity In The United States
Percentage Deviations From Trend
(Initial And Final Portions Only)

INITIAL PORTION

	1899	1900	1901
Jan	- 1.8	- 0.4	- 4.3
Feb	- 2.8	- 1.7	- 1.2
Mar	- 0.2	- 3.0	- 0.2
Apr	+ 0.2	- 3.8	+ 4.1
May	+ 2.4	- 4.9	+ 6.2
Jun	+ 2.8	- 9.1	+ 6.5
Jul	+ 4.1	-10.2	+ 7.2
Aug	+ 3.8	-11.6	+ 5.0
Sep	+ 3.4	-11.2	+ 4.2
Oct	+ 3.4	-10.7	+ 1.7
Nov	+ 1.4	-10.3	+ 1.8
Dec	+ 3.8	- 7.1	- 0.2

FINAL PORTION

	1937	1938	1939
Jan	+ 1.4	-28.6	-12.3
Feb	+ 3.3	-26.9	-11.3
Mar	+ 5.1	-26.5	-10.5
Apr	+ 6.0	-27.7	-13.0
May	+ 6.5	-28.1	-13.0
Jun	+ 3.4	-27.9	-10.2
Jul	+ 2.6	-23.4	- 9.8
Aug	+ 0.6	-19.7	- 7.8
Sep	- 3.0	-17.8	- 2.3
Oct	-14.2	-16.8	+ 3.8
Nov	-24.3	-12.5	+ 6.0
Dec	-29.9	-13.4	+ 6.2

TABLE II

41–MONTH HARMONIC ANALYSES OF FIRST AND TWELFTH SEGMENTS OF SERIES

(a)	(b) θ*	(c)	(d)	(e) 1st Segment			(f) 12th Segment		
Month	(a÷41)	Sin θ	Cos θ	Y#	Y Sin θ	Y Cos θ	Y#	Y Sin θ	Y Cos θ
1	2.44%	0.153	0.988	-1.8	-.28	-1.78	-6.1	-.93	-6.03
2	4.88	.302	.953	-2.8	-.85	-2.67	-3.7	-1.12	-3.53
3	7.32	.444	.896	-0.2	-.09	-0.18	-3.5	-1.55	-3.14
4	9.76	.576	.817	+0.2	+.12	+0.16	-1.7	-.98	-1.39
5	12.20	.694	.720	+2.4	+1.67	+1.73	+2.2	+1.53	+1.58
6	14.63	.795	.606	+2.8	+1.70	+1.70	+1.4	+1.11	+.85
7	17.07	.878	.478	+4.1	+3.60	+1.96	+3.3	+2.90	+1.58
8	19.51	.941	.338	+3.8	+3.58	+1.28	+5.1	+4.80	+1.72
9	21.96	.982	.190	+3.4	+3.34	+0.65	+6.0	+5.89	+1.14
10	24.39	.999	.039	+3.4	+3.40	+0.13	+6.5	+6.49	+.25
11	26.83	.994	-.115	+1.4	+1.39	-0.16	+3.4	+3.38	-.39
12	29.27	.964	-.265	+3.8	+3.66	-1.01	+2.6	+2.51	-.69
13	31.71	-.912	-.409	-0.4	-.36	+0.16	+0.6	+.55	-.25
14	34.15	-.839	-.543	-1.7	-1.43	+0.92	-3.0	-2.52	+1.63
15	36.59	.746	-.666	-3.0	-2.24	+2.00	-14.2	-10.59	+9.46
16	39.02	.636	-.771	-3.8	-2.42	+2.93	-24.3	-15.45	+18.74
17	41.46	.511	-.859	-4.9	-2.50	+4.21	-29.9	-15.28	+25.68
18	43.90	.374	-.927	-9.1	-3.40	+8.44	-28.6	-10.70	+26.51
19	46.34	.228	-.973	-10.2	-2.33	+9.92	-26.9	-6.13	+26.17
20	48.78	.076	-.997	-11.6	-.88	+11.57	-26.5	-2.01	+26.42
21	51.22	-.076	-.997	-11.2	+.85	+11.17	-27.7	+2.11	+27.62
22	53.66	-.228	-.973	-10.7	+2.44	+10.41	-28.1	+6.41	+27.34
23	56.10	-.374	-.927	-10.3	+3.85	+9.55	-27.9	+10.43	+25.86
24	58.54	-.511	-.859	-7.1	+3.63	+6.10	-23.4	+11.96	+20.10
25	60.98	-.636	-.771	-4.3	+2.73	+3.32	-19.7	+12.53	+15.19
26	63.41	-.746	-.666	-1.2	+.90	+.80	-17.8	+13.28	+11.85
27	65.85	-.839	-.543	+0.2	-.17	-.11	-16.8	+14.10	+9.12
28	68.29	-.912	-.409	+4.1	-3.74	-1.68	-12.5	+11.40	+5.11
29	70.73	-.964	-.265	+6.2	-5.98	-1.64	-13.4	+12.92	+3.55
30	73.17	-.994	-.115	+6.5	-6.46	-0.75	-12.3	+12.23	+1.41
31	75.61	-.999	.039	+7.2	-7.19	+0.28	-11.3	+11.29	-.44
32	78.05	-.982	.190	+5.0	-4.91	+0.95	-10.5	+10.31	-2.00
33	80.49	-.941	.338	+4.2	-3.95	+1.42	-13.0	+12.23	-4.39
34	82.93	-.878	.478	+1.7	-1.49	+0.81	-13.0	+11.41	-6.21
35	85.37	-.795	.606	+1.8	-1.43	+1.09	-10.2	+8.11	-6.18
36	87.80	-.694	.720	-0.2	+.14	-0.14	-9.8	+6.80	-7.06
37	90.24	-.576	.817	+1.0	-.58	+0.82	-7.8	+4.49	-6.37
38	92.68	-.444	.896	+1.7	-.75	+1.52	-2.3	+1.02	-2.06
39	95.12	-.302	.953	+2.4	-.72	+2.29	+3.8	-1.15	+3.62
40	97.56	-.153	.988	+2.2	-.34	+2.17	+6.0	-.92	+5.93
41	100.00	-	1.000	+2.6	0	+2.60	+6.2	0	+6.20
			Total		-16.96 (A)	92.94 (B)		132.86 (A)	254.50 (B)

*Expressed as Percentages of 360°

From Table I.

CHART I

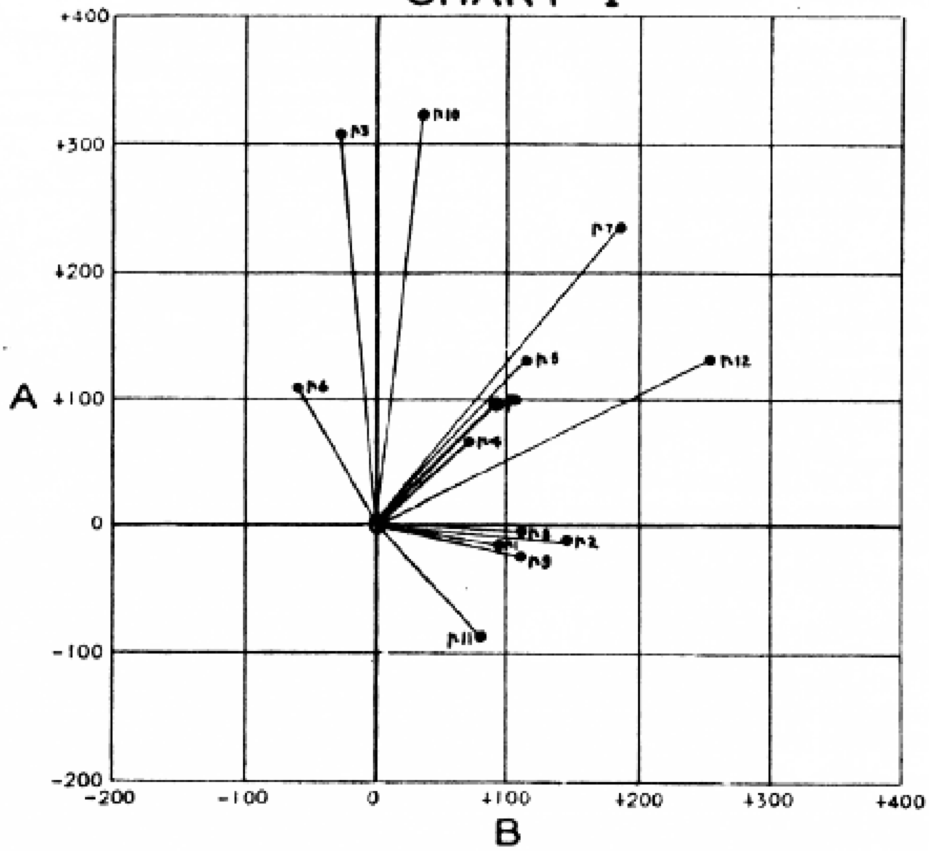

TABLE OF EXPONENTIALS

Values of e^n, From n = 1.0 to n = 10.9

(e = 2.71828)

n	.0	0.1	0.2	0.3	0.4	0.51	0.6	0.7	0.8	0.9
1	2.72	3.00	3.32	3.67	4.06	4.48	4.95	5.47	6.05	6.69
2	7.39	8.17	9.02	9.97	11.0	12.2	13.5	14.9	16.4	18.2
3	20.1	22.2	24.5	27.1	30.0	33.1	36.6	40.4	44.7	49.4
4	54.6	60.3	66.7	73.7	81.4	90.0	99.5	110	122	134
5	148	164	181	200	221	245	270	299	330	365
6	403	446	493	545	602	665	735	812	898	992
7	1097	1212	1339	1480	1636	1808	1998	2208	2441	2698
8	2981	3294	3639	4024	4447	4915	5431	6003	6635	7333
9	8103	8954	9894	10940	12090	13360	14760	16320	18040	19930
10	22030	24340	26890	29740	32860	36320	40130	44360	49030	54180

TABLE III

Tabulation Of Results Of Harmonic Analysis Of Twelve Segments of Series

Segment	A (From Table II)	B (From Table II)	$A^2 - B^2$
1	- 16.96	+ 92.94	8,925.49
2	- 11.05	+ 143.68	20,766.04
3	+ 308.14	- 29.18	95,801.73
4	+ 66.13	+ 70.25	9,308.24
5	+ 131.35	+ 115.42	30,574.60
6	+ 108.94	- 63.35	15,881.15
7	+ 236.80	+ 184.75	90,206.80
8	- 4.90	+ 110.45	12,223.21
9	- 24.97	+ 109.94	12,710.30
10	+ 321.10	+ 34.98	104,328.81
11	- 86.42	+ 79.24	13,747.39
12	+ 132.86	+ 254.50	82,422.03
TOTAL	+1,161.02	+1,103.62	496,895.79
AVERAGE	+ 96.75	+ 91.97	41,407.98

Average Amplitude = $\sqrt{96.75^2 + 91.97^2}$ = 133.49

Quadratic Mean of Twelve Individual Amplitudes = $\sqrt{41,407.98}$ = 203.49

FOOTNOTES

1The values of sine and cosine used in Table II were taken from the trigonometric table shown in the Appendix. This table shows directly the values of sine θ and cosine θ for values of θ expressed as percentages of 360°. It is particularly useful in harmonic analysis, as it eliminates the step of converting to degrees, minutes, and seconds, angles which are much more readily expressed in percentage form.

2Chart I is essentially a "harmonic dial", differing from it only in using rectangular coordinates in place of polar. The construction of the chart is useful in illustrating the implications of the test and in presenting visually the results of the test, but, in practice, the final probabilities may be determined directly from the results computed as in Table III.

3The distance $Op_1 = \sqrt{A_1^2 + B_1^2}$, and therefore does not equal the amplitude of the fitted sinusoid, but is a multiple of that amplitude. This amplitude "a_1" may be derived, if desired, from the equation $a_1 = \frac{2}{41}\sqrt{A_1^2 + B_1^2}$.

4As an alternative approach, Bartels' also assumes P to be the true point center, and computes the probability that the center of the twelve points could be as far from P as O is. As compared with the method outlined above, such an approach gives an even smaller probability that the cycle is of chance origin.